Warsaw University of Technology

Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

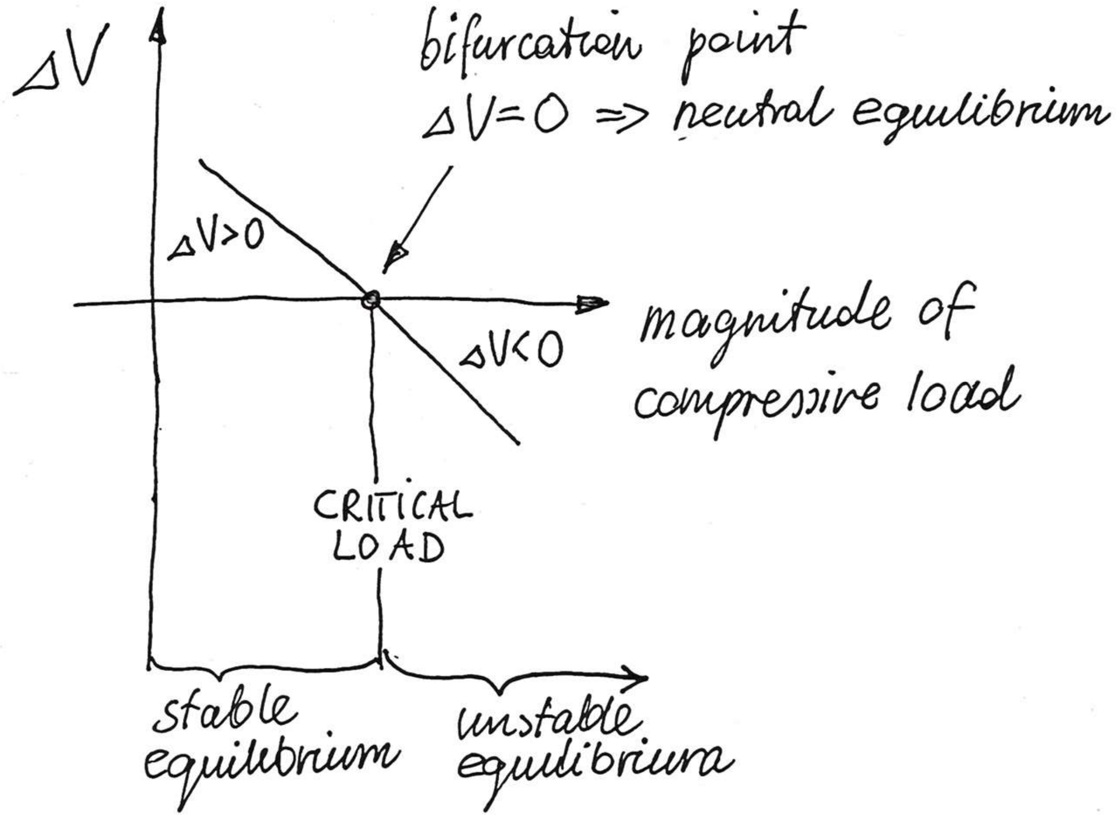
Finite element method 2 (FEM 2)

Buckling analysis

11.2021

BUCKLING ANALYSIS

The stability of a linear structure depends on the change of total potential energy due to a very small disturbance from the equilibrium state.



(Taylor series)

$$\Delta V = \delta V + \frac{1}{2!}\delta^2 V + \frac{1}{3!}\delta^3 V$$

3rd variation (neglected)

2nd variation included in the Lagrange-Dirichlet criterion

1st variation $\delta V = 0$ - necessary condition of equilibrium $V \to min$

For a very small disturbance of the displacement vector:

$$\underset{1\times N}{\lfloor dq \rfloor} = \lfloor dq_1, dq_2, \ldots, dq_N \rfloor$$

The change of V:

$$\Delta V = \sum_{i=1}^{N} \underbrace{\frac{\partial V}{\partial q_i}}_{\substack{0 \\ \left(\underset{N\times N}{[K]} \underset{N\times 1}{\{q\}} = \underset{N\times 1}{\{F\}} \right)}} dq_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 V}{\partial q_i \partial q_j} \cdot dq_i \, dq_j =$$

LINEAR PART

NONLINEAR PART

$$= 0 + \frac{1}{2} \underset{1\times N}{\lfloor dq \rfloor} \underset{N\times N}{\left[\frac{\partial^2 V}{\partial q_i \partial q_j} \right]} \cdot \underset{N\times 1}{\{dq\}}$$

$$\left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right]_{N\times N} = \begin{bmatrix} \frac{\partial^2 V}{\partial q_1^2} & \frac{\partial^2 V}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 V}{\partial q_1 \partial q_N} \\ \frac{\partial^2 V}{\partial q_2 \partial q_1} & \frac{\partial^2 V}{\partial q_2^2} & & \frac{\partial^2 V}{\partial q_2 \partial q_N} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 V}{\partial q_N \partial q_1} & \frac{\partial^2 V}{\partial q_N \partial q_2} & \cdots & \frac{\partial^2 V}{\partial q_N \partial q_N} \end{bmatrix}$$

$\Delta V = 0$ for a neutral (or critical) equilibrium :

$$\underbrace{\left[\frac{\partial^2 V(\lambda_*)}{\partial q_i \, \partial q_j}\right]}_{N \times N} \cdot \underbrace{\{dq\}}_{N \times 1} = \underbrace{\{0\}}_{N \times 1}$$

where :

λ_* - load multiplier , for which $\Delta V = 0$

finally :

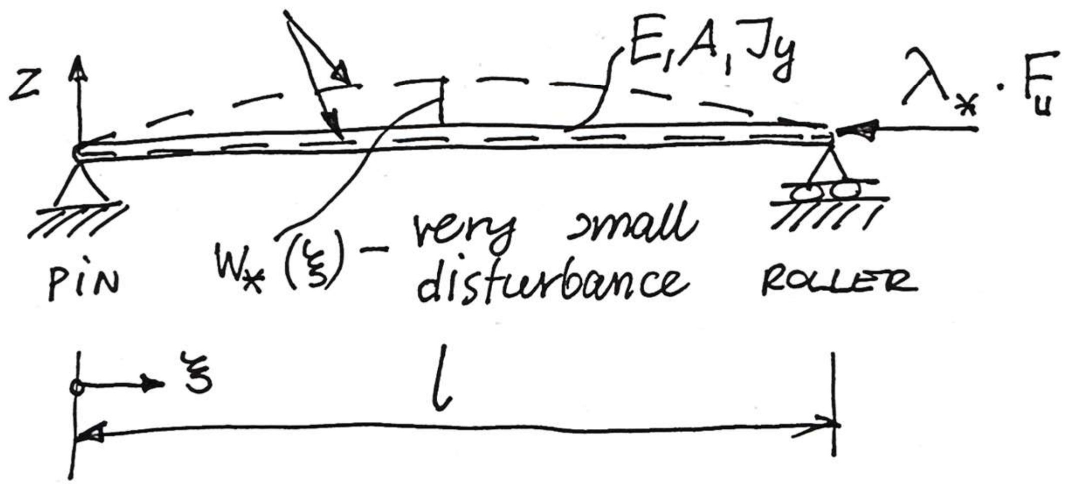
$$\det\left(\underbrace{\left[\frac{\partial^2 V(\lambda_*)}{\partial q_i \, \partial q_j}\right]}_{N \times N}\right) = 0$$

(eigen problem)

EXAMPLE. FIND THE CRITICAL LOAD AND BUCKLING MODES FOR A COLUMN. USE ONE FINITE ELEMENT.

F_u - unit load , $F_u = 1N$

neutral equilibrium - two deformations are possible for the critical load ($\Delta V = 0$)



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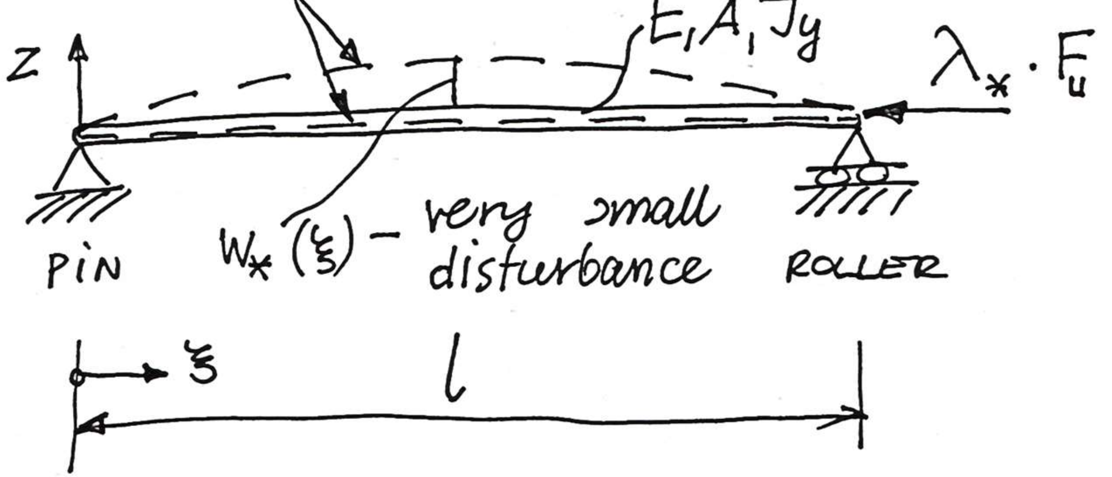
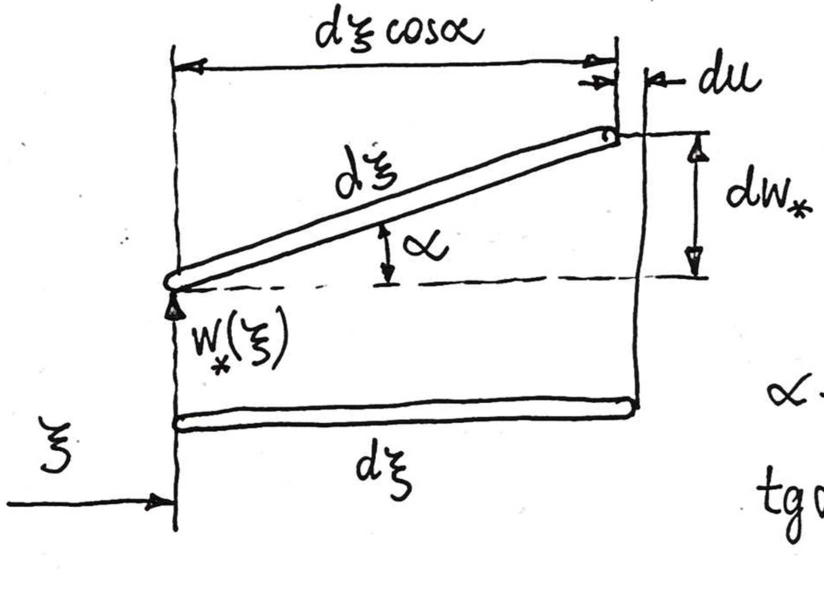


$F_u = 1N$ – unit load



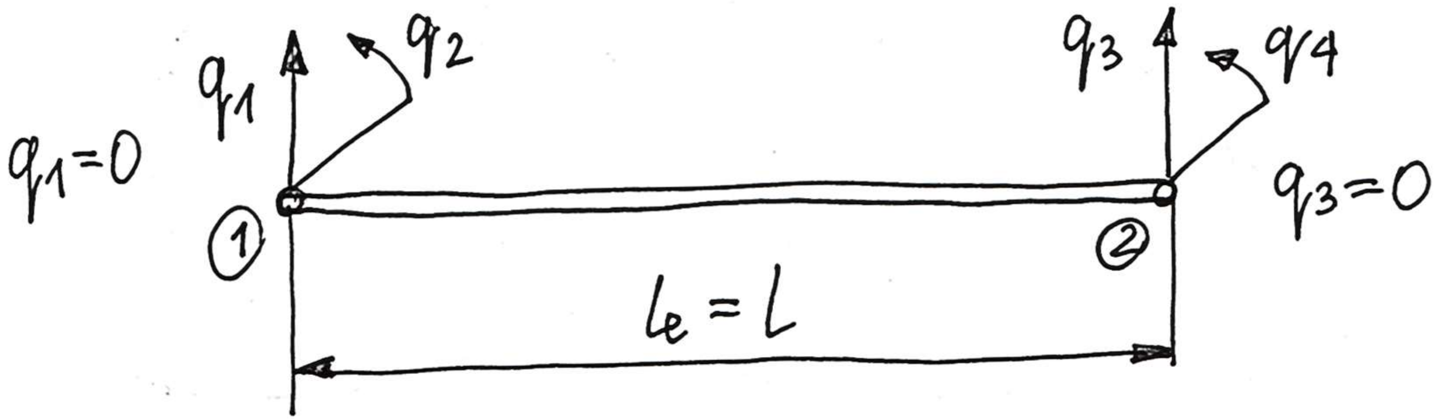
α – small angle

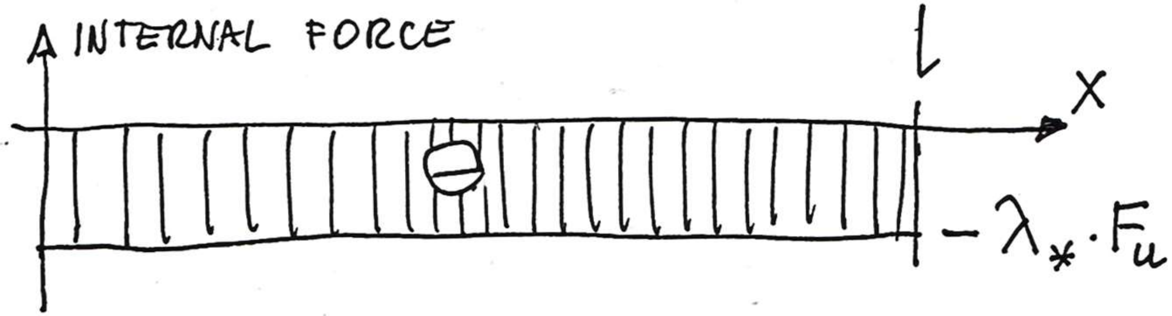
$$\operatorname{tg}\alpha \cong \alpha = \frac{dw_*}{d\xi}$$

$$du = d\xi - d\xi\cdot\cos\alpha = d\xi\,(1-\cos\alpha) =$$

$$= \left| \begin{array}{ll} \cos 2\varphi = \cos^2\varphi - \sin^2\varphi \;\Rightarrow & \cos 2\varphi = 1 - 2\sin^2\varphi \\ & 1-\cos 2\varphi = 2\sin^2\varphi \\ \cos^2\varphi = 1-\sin^2\varphi & \\ & 2\varphi = \alpha \;\Rightarrow\; \varphi = \frac{\alpha}{2} \end{array} \right| =$$

$$= d\xi\cdot 2\sin^2\left(\frac{\alpha}{2}\right) \cong d\xi\cdot 2\cdot\frac{\alpha^2}{4} = \frac{1}{2}\left(\frac{dw_*}{d\xi}\right)^2 d\xi$$


q_1
q_2
q_3
q_4
$q_1=0$
$q_3=0$
1
2
$l_e = L$


INTERNAL FORCE
L
X
−
$-\lambda_* \cdot F_u$

Nonlinear part of strain energy:

$$\Delta U^{NL} = \frac{1}{2} \int_0^l E J_y \left(\frac{d^2 W_*}{d\xi^2} \right)^2 d\xi$$

We consider only the nonlinear part of ΔV

so the strain energy of axial loading

on the axial displacement is neglected:

Nonlinear part of potential energy of loading :

$$\Delta W^{NL} = \int_0^l (\lambda_* \cdot F_u)\, du = \frac{1}{2}\int_0^l (\lambda_* \cdot F_u)\left(\frac{dw_*}{d\xi}\right)^2 d\xi$$

We consider only the nonlinear part of ΔV

so the potential energy of axial loading

on the axial displacement is neglected :

$$w_*(\xi) = \lfloor N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi) \rfloor \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\frac{dw_*}{d\xi} = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi}, \frac{dN_3}{d\xi}, \frac{dN_4}{d\xi} \right\rfloor \cdot \underset{4\times 1}{\{q\}_e} = \underset{1\times 4}{\lfloor N' \rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$

$$\frac{d^2w_*}{d\xi^2} = \left\lfloor \frac{d^2N_1}{d\xi^2}, \frac{d^2N_2}{d\xi^2}, \frac{d^2N_3}{d\xi^2}, \frac{d^2N_4}{d\xi^2} \right\rfloor \cdot \underset{4\times 1}{\{q\}_e} = \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$

$$\Delta V^{NL} = \Delta U^{NL} - \Delta W^{NL} =$$

$$= \frac{1}{2}\int_0^l EJ_y \cdot \underset{\Uparrow \atop \underset{1\times4}{\lfloor q \rfloor_e} \cdot \underset{4\times1}{\{N''\}}}{\frac{d^2 w_*}{d\xi^2}} \cdot \underset{\nwarrow \atop \underset{1\times4}{\lfloor N'' \rfloor} \cdot \underset{4\times1}{\{q\}_e}}{\frac{d^2 w_*}{d\xi^2}} \, d\xi \; - \frac{1}{2}\int_0^l (\lambda_* \cdot F_u) \cdot \underset{\Uparrow \atop \underset{1\times4}{\lfloor q \rfloor_e} \cdot \underset{4\times1}{\{N'\}}}{\frac{dw_*}{d\xi}} \cdot \underset{\nwarrow \atop \underset{1\times4}{\lfloor N' \rfloor} \cdot \underset{4\times1}{\{q\}_e}}{\frac{dw_*}{d\xi}} \, d\xi =$$

$$= \frac{1}{2} \underset{1\times4}{\lfloor q \rfloor_e} \underbrace{\int_0^l EJ_y \underset{4\times1}{\{N''\}} \underset{1\times4}{\lfloor N'' \rfloor} \, d\xi}_{\text{stiffness matrix } \underset{4\times4}{[k]_e}} \cdot \underset{4\times1}{\{q\}_e} - \frac{1}{2} \underset{1\times4}{\lfloor q \rfloor_e} \cdot \lambda_* \cdot \underbrace{\int_0^l \underset{4\times1}{\{N'\}} \cdot F_u \cdot \underset{1\times4}{\lfloor N' \rfloor} d\xi}_{\text{prestress matrix } \underset{4\times4}{[k_\sigma]_e}} \cdot \{q\}_e =$$

$$= \frac{1}{2} \underset{1\times4}{\lfloor q \rfloor_e} \left(\underset{4\times4}{[k]_e} - \lambda_* \underset{4\times4}{[k_\sigma]_e} \right) \cdot \underset{4\times1}{\{q\}_e}$$

stiffness matrix

$$\underset{4\times4}{[k]_e} = \frac{2EJ_y}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

prestress matrix

$$\underset{4\times4}{[k_\sigma]_e} = \frac{F_u}{30l}\cdot\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}$$

FOR THE CRITICAL LOAD: $\Delta V = 0$

$$q_1 = q_3 = 0 \quad \Rightarrow \quad \Big(\underset{2\times 2}{[K]_e} - \lambda_* \underset{2\times 2}{[K_\sigma]_e} \Big) \cdot \underset{2\times 1}{\{q\}_e} = \underset{2\times 1}{\{0\}}$$

$$\left(\frac{2EJ_y}{l^3} \begin{bmatrix} 2l^2 & l^2 \\ l^2 & 2l^2 \end{bmatrix} - \lambda_* \cdot \frac{F_u}{30l} \cdot \begin{bmatrix} 4l^2 & -l^2 \\ -l^2 & 4l^2 \end{bmatrix} \right) \cdot \underset{2\times 1}{\{q\}_e} = \underset{4\times 1}{\{0\}}$$

$$\det \Big(\underset{2\times 2}{[K]_e} - \lambda_* \underset{2\times 2}{[K_\sigma]_e} \Big) = 0$$

$$\det \left(\frac{2EJ_y}{l^3} \begin{bmatrix} 2l^2 & l^2 \\ l^2 & 2l^2 \end{bmatrix} - \lambda_* \cdot \frac{F_u}{30l} \cdot \begin{bmatrix} 4l^2 & -l^2 \\ -l^2 & 4l^2 \end{bmatrix} \right) = 0$$

after introducing a new constant: $\lambda = \lambda_* \cdot \frac{F_u \cdot L^2}{60 E J_y}$

$$\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} 2-4\lambda & 1+\lambda \\ 1+\lambda & 2-4\lambda \end{bmatrix} \right) = 0$$

$$(2-4\lambda)^2 - (1+\lambda)^2 = 0$$

$$4 - 16\lambda + 16\lambda^2 - 1 - 2\lambda - \lambda^2 = 0$$

$$15\lambda^2 - 18\lambda + 3 = 0$$

$$\Delta = (-18)^2 - 4 \cdot 15 \cdot 3 = 324 - 180 = 144$$

$$\sqrt{\Delta} = 12$$

$$\lambda_1 = \frac{18-12}{30} = \frac{1}{5} \qquad \lambda_2 = \frac{18+12}{30} = 1$$

$$\lambda_{*1} = \lambda_1 \cdot \frac{60\,EJ_y}{F_u \cdot l^2} = \frac{12\,EJ_y}{F_u \cdot l^2}$$

$$\lambda_{*2} = \lambda_2 \cdot \frac{60\,EJ_y}{F_u \cdot l^2} = \frac{60\,EJ_y}{F_u \cdot l^2}$$

FIRST CRITICAL LOAD $\quad F_{CRIT\,1} = \lambda_{*1} \cdot F_u = \dfrac{12\,EJ_y}{l^2}$

SECOND CRITICAL LOAD $\quad F_{CRIT\,2} = \lambda_{*2} \cdot F_u = \dfrac{60\,EJ_y}{l^2}$

ANALYTICAL SOLUTION – EULER'S CRITICAL FORCE : $\overline{F}_{CRIT_i} = i^2 \dfrac{\pi^2 E J_y}{l^2}$

$$\overline{F}_{CRIT_1} = \frac{\pi^2 E J_y}{l^2} = 9.87 \frac{E J_y}{l^2}$$

$$\overline{F}_{CRIT\,2} = \frac{4\pi^2 E J_y}{l^2} = 39.48 \frac{E J_y}{l^2}$$

$$\Delta F_{CRIT_1} = \frac{F_{CRIT_1} - \overline{F}_{CRIT_1}}{\overline{F}_{CRIT_1}} = 22\%$$

$$\Delta F_{CRIT_2} = \frac{F_{CRIT_2} - \overline{F}_{CRIT_2}}{\overline{F}_{CRIT_2}} = 52\%$$

BUCKLING MODES :

$$\begin{bmatrix} 2-4\lambda_i & 1+\lambda_i \\ 1+\lambda_i & 2-4\lambda_i \end{bmatrix} \begin{Bmatrix} q_2(\lambda_i) \\ q_4(\lambda_i) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

two linearly dependent equations

1st :

$$(2-4\lambda_i)\cdot q_2(\lambda_i) + (1+\lambda_i)\cdot q_4(\lambda_i) = 0$$

$$q_4(\lambda_i) = -\frac{2-4\lambda_i}{1+\lambda_i}\cdot q_2(\lambda_i)$$

LETS ASSUME : $q_2(\lambda_1) = q_2(\lambda_2) = a$

(any angle)

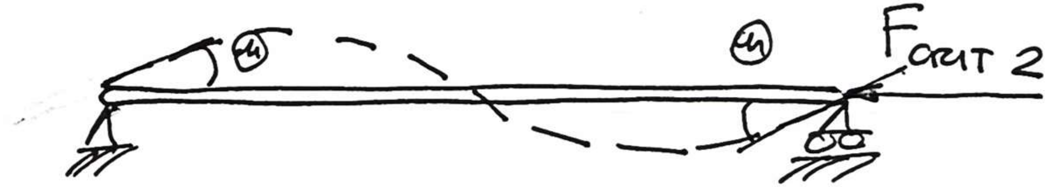
$$q_4(\lambda_1) = -\frac{2-4\cdot\frac{1}{5}}{1+\frac{1}{5}}\cdot u = -u$$

$$q_4(\lambda_2) = -\frac{2-4\cdot 1}{1+1}\cdot u = u$$

1ST BUCKLING MODE

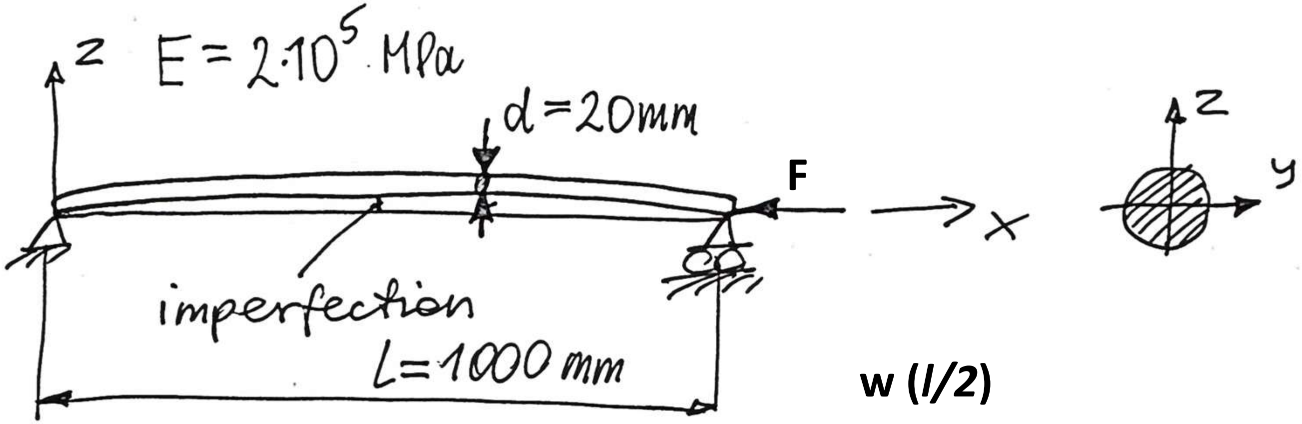
$$F_{CRIT\,1} = \frac{12EJ_y}{l^2}$$

2ND BUCKLING MODE



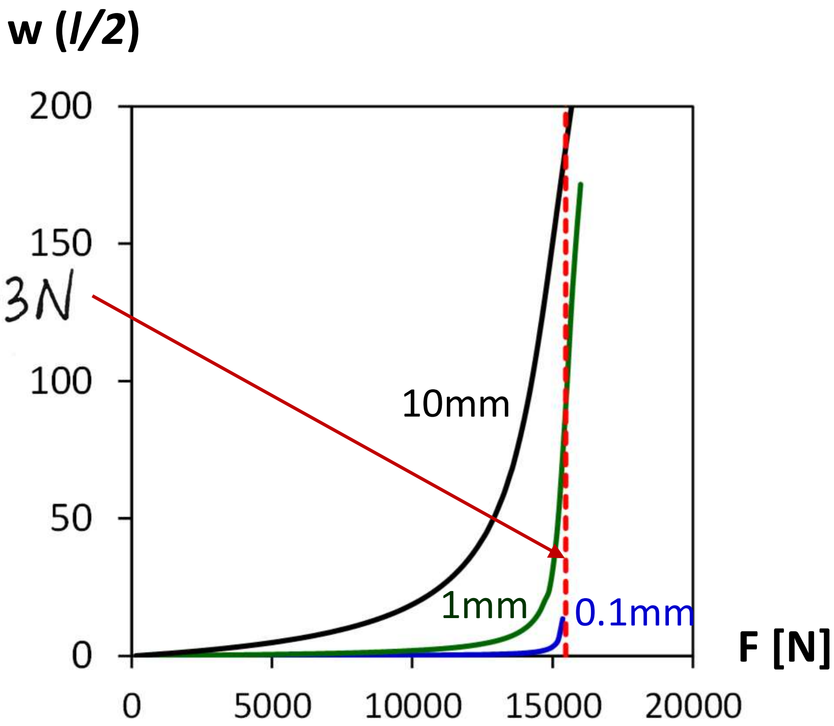
$$F_{CRIT\,2} = \frac{60EJ_y}{l^2}$$

NONLINEAR SOLUTION

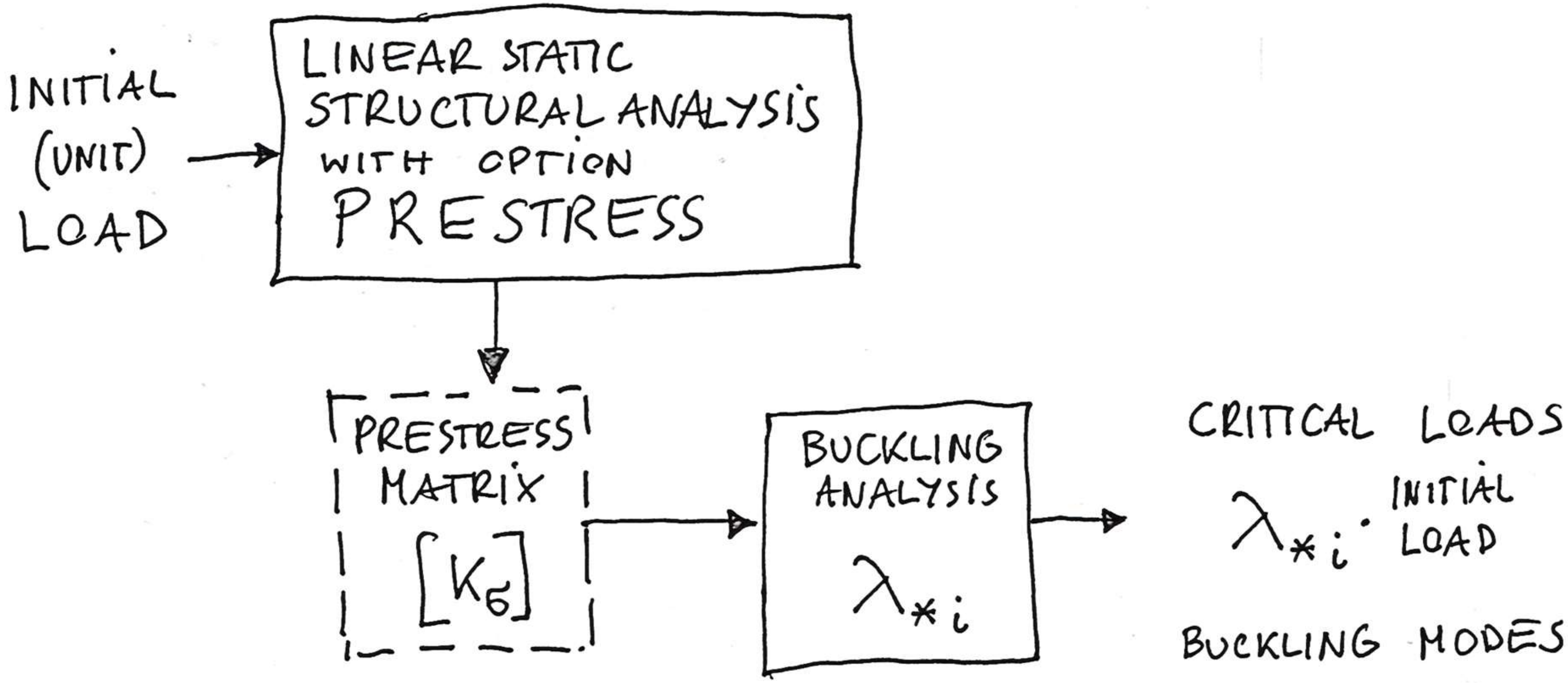


$$J_y = \frac{\pi d^4}{64}$$

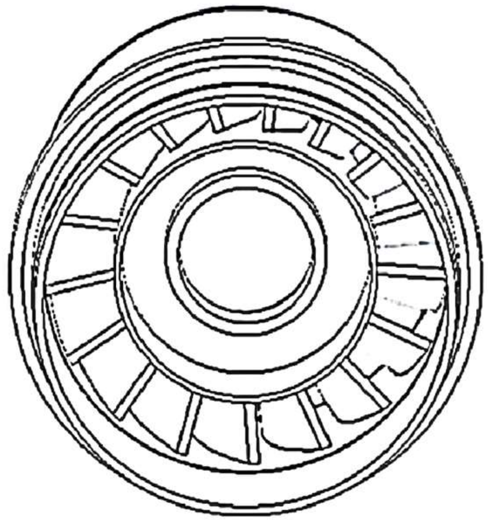
$$F_{CRIT\,1} = \frac{\pi^2 E J_y}{l^2} = \frac{\pi^3 E d^4}{64\, l^2} = 15503N$$



BUCKLING ANALYSIS



EXAMPLE . BUCKLING OF THE NOZZLE GUIDE VANE
DUE TO TEMPERATURE INCREMENT



$\Delta T_{CRIT} = ?$

surface load
$p = const$

$$\Delta T_{CRIT} = \lambda_{*1} \cdot \underbrace{\Delta T_U}_{1°C}$$

$\uparrow$ FIRST CRITICAL LOAD

$$p_{CRIT} = \lambda_{*1} \cdot p \quad \text{but} \quad p = const \Rightarrow \lambda_{*1} = 1$$

ITERATIVE SOLUTION:

STEP	INITIAL LOAD	PRESTRESS MATRIX	BUCKLING ANALYSIS
1	$\Delta T_u^1 = 1°C,\ p$	$[K_\sigma]_1$	λ_{*1}^1
2	$\Delta T_u^2 = \lambda_{*1}^1 \cdot 1°C,\ p$	$[K_\sigma]_2$	λ_{*1}^2
⋮	⋮	⋮	⋮
j	$\Delta T_u^j = \lambda_{*1}^{j-1} \cdot 1°C,\ p$	$[K_\sigma]_j$	λ_{*1}^j
⋮	⋮	⋮	⋮
LAST	⋮	⋮	$\lambda_{*1}^{LAST} \cong 1.0$